

MODULES WHICH ARE COINVARIANT UNDER AUTOMORPHISMS OF THEIR PROJECTIVE COVERS

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ABSTRACT. In this paper we study modules coinvariant under automorphisms of their projective covers. We first provide an alternative, and in fact, a more succinct and conceptual proof for the result that a module M is invariant under automorphisms of its injective envelope if and only if given any submodule N of M , any monomorphism $f : N \rightarrow M$ can be extended to an endomorphism of M and then, as a dual of it, we show that over a right perfect ring, a module M is coinvariant under automorphisms of its projective cover if and only if for every submodule N of M , any epimorphism $\varphi : M \rightarrow M/N$ can be lifted to an endomorphism of M .

1. INTRODUCTION

Modules invariant or coinvariant under automorphisms of their covers or envelopes have been recently introduced in [7]. Recall that a class \mathcal{X} of right modules over a ring R , closed under isomorphisms, is called an *enveloping class* if for any right R -module M , there exists a homomorphism $u : M \rightarrow X(M)$, with $X(M) \in \mathcal{X}$, such that any other morphism from M to a module in \mathcal{X} factors through u and, moreover, whenever u has a factorization $u = h \circ u$, then h must be an automorphism. This morphism u is called the \mathcal{X} -*envelope* of M . And this envelope is a *monomorphic envelope* if, in addition, u is a monomorphism. Dually, \mathcal{X} is called a *covering class* if for any right R -module M , there exists a homomorphism $p : X(M) \rightarrow M$ such that any other homomorphism from an object of \mathcal{X} to M factors through p and moreover, whenever p factors as $p = p \circ h$, h must be an automorphism. This morphism p is called the \mathcal{X} -*cover* of M and this cover is said to be an *epimorphic cover* if p is an epimorphism.

A module M having a monomorphic \mathcal{X} -envelope $u : M \rightarrow X(M)$ (resp. epimorphic \mathcal{X} -cover $p : X(M) \rightarrow M$) is said to be *invariant under φ* (resp. *coinvariant under φ*), $\varphi : X(M) \rightarrow X(M)$, if there exists an endomorphism $f : M \rightarrow M$ such that $u \circ f = \varphi \circ u$ (resp. $f \circ p = p \circ \varphi$).

A module M having a monomorphic \mathcal{X} -envelope $u : M \rightarrow X(M)$ (resp. epimorphic \mathcal{X} -cover $p : X(M) \rightarrow M$) is said to be \mathcal{X} -*automorphism invariant* (resp. \mathcal{X} -*automorphism coinvariant*) if M is invariant (resp. coinvariant) under each automorphism $\varphi : X(M) \rightarrow X(M)$.

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If a module M is invariant (resp. coinvariant) under each endomorphism $\varphi : X(M) \rightarrow X(M)$, then M is called \mathcal{X} -endomorphism invariant (resp. \mathcal{X} -endomorphism coinvariant).

When \mathcal{X} is the class of injective modules, \mathcal{X} -automorphism invariant modules are usually just called *automorphism-invariant* modules and \mathcal{X} -endomorphism invariant modules are called *quasi-injective* modules. When \mathcal{X} is the class of projective modules, \mathcal{X} -automorphism coinvariant modules are called *automorphism-coinvariant* modules and \mathcal{X} -endomorphism coinvariant modules are called *quasi-projective* modules.

Automorphism-invariant modules have been studied extensively in [1, 4, 5, 8, 9, 10, 14, 16]. On the other hand, it was proved in [5] that a module M is automorphism-invariant if and only if any monomorphism from a submodule N of M to M extends to an endomorphism of M . The main goal of this note is to give a new and more conceptual proof of this result which allows to dualize it to automorphism-coinvariant modules.

Throughout this note, all rings will be associative rings with identity and ‘module’ will mean a unitary right module unless otherwise stated. We refer to [2, 3] for any undefined notion used along the text.

2. MAIN RESULTS

We begin by noting an important structural result from [11] which will be of crucial importance throughout.

Theorem 2.1. [11] *Let \mathcal{X} be an enveloping (resp., covering) class of modules. If $u : M \rightarrow X$ is a monomorphic \mathcal{X} -envelope (resp., $p : X \rightarrow M$ is an epimorphic cover) of a module M such that M is \mathcal{X} -automorphism invariant (resp., \mathcal{X} -automorphism coinvariant) and $\text{End}(X)/J(\text{End}(X))$ is a von Neumann regular right self-injective ring and idempotents lift modulo $J(\text{End}(X))$, then $\text{End}(M)/J(\text{End}(M))$ is also a von Neumann regular ring and idempotents in $\text{End}(M)/J(\text{End}(M))$ lift to idempotents in $\text{End}(M)$. Moreover, M admits a decomposition $M = N \oplus L$ such that:*

- (i) $\text{End}(N)/J(\text{End}(N))$ is a Boolean ring.
- (ii) Each element of $\text{End}(L)$ is the sum of two units and consequently, L is \mathcal{X} -endomorphism invariant (resp., \mathcal{X} -endomorphism coinvariant).
- (iii) Both $\text{Hom}_R(N, L)$ and $\text{Hom}_R(L, N)$ are contained in $J(\text{End}(M))$.

In particular, $\text{End}(M)/J(\text{End}(M))$ is the direct product of a Boolean ring and a right self-injective von Neumann regular ring.

Recall that a module M is called *pseudo-injective* if given any submodule A of M , any monomorphism $f : A \rightarrow M$ can be extended to an endomorphism of M (see [12, 17]). We will first give a new proof showing that automorphism-invariant modules coincide with pseudo-injective modules.

Throughout this section, we will follow notations as in [7] to write elements in direct product of rings $R_1 \times R_2$ as $a_1 \times a_2$ where $a_1 \in R_1$ and $a_2 \in R_2$. To write an element in $R/J(R)$, we will use the notation \bar{a} where $a \in R$.

Theorem 2.2. [5] *A module M is automorphism-invariant if and only if it is pseudo-injective.*

Proof. Let M be an automorphism-invariant module. Let N be a submodule of M and $f : N \rightarrow M$ be a monomorphism. Call $E = E(M)$. It is well-known that $\text{End}(E)/J(\text{End}(E))$ is von Neumann regular, right self-injective and idempotents lift modulo $J(\text{End}(E))$. Since injective modules are obviously automorphism-invariant, from Theorem 2.1, we have that $E = E_1 \oplus E_2$, where $\text{Hom}(E_1, E_2), \text{Hom}(E_2, E_1) \subseteq J(\text{End}(E))$ and

$$\text{End}(E)/J(\text{End}(E)) = \text{End}(E_1)/J(\text{End}(E_1)) \times \text{End}(E_2)/J(\text{End}(E_2))$$

such that $\text{End}(E_1)/J(\text{End}(E_1))$ is a Boolean ring and each element in the ring $\text{End}(E_2)/J(\text{End}(E_2))$ is the sum of two units. Call $S = \text{End}(E)$, $S_1 = \text{End}(E_1)$ and $S_2 = \text{End}(E_2)$. Let $v : E(N) \rightarrow E$ be the inclusion and $p : E \rightarrow E(N)$, an epimorphism that splits v . Then $e = v \circ p \in S$ is an idempotent such that $E(N) = eE$. By injectivity, f extends to a monomorphism $g : E(N) \rightarrow E$. This monomorphism g splits as $E(N)$ is injective. So there exists an epimorphism $\delta : E \rightarrow E(N)$ such that $\delta \circ g = 1_{E(N)}$. Call $h = g \circ p \in S$. We claim that $\bar{h}|_{\bar{e}\bar{S}} : \bar{e}\bar{S} \rightarrow \bar{S}$ is a monomorphism. Let $x \in S$ such that $\bar{e}\bar{x} \in \ker(\bar{h})$. This means that $h \circ e \circ x = g \circ p \circ e \circ x$ has essential kernel. And, as g is a monomorphism, we deduce that $p \circ e \circ x = p \circ v \circ p \circ x = p \circ x$ also has essential kernel. So, $e \circ x = v \circ p \circ x$ has essential kernel too. This shows that $\bar{e}\bar{x} = 0$ and thus, $\bar{h}|_{\bar{e}\bar{S}}$ is monic.

As $\bar{S} = \bar{S}_1 \times \bar{S}_2$, there exist idempotents $\bar{e}_1 \in \bar{S}_1$ and $\bar{e}_2 \in \bar{S}_2$ such that $\bar{e} = \bar{e}_1 \times \bar{e}_2 \in \bar{S}_1 \times \bar{S}_2$ and homomorphisms $\bar{h}_1 : \bar{S}_1 \rightarrow \bar{S}_1$ and $\bar{h}_2 : \bar{S}_2 \rightarrow \bar{S}_2$ such that $\bar{h} = \bar{h}_1 \times \bar{h}_2$. Moreover, $\bar{h}_i|_{\bar{e}_i\bar{S}_i} : \bar{e}_i\bar{S}_i \rightarrow \bar{S}_i$ is a monomorphism and $\bar{h}_i|_{(1-\bar{e}_i)\bar{S}_i} = 0$ for $i = 1, 2$. As $\text{Im}(\bar{h}) \cong \bar{e}\bar{S}$, it is a direct summand of \bar{S} . So there exists an idempotent $\bar{e}' \in \bar{S}$ such that $\text{Im}(\bar{h}) = \bar{e}'\bar{S}$. And again, $\bar{e}' = \bar{e}'_1 \times \bar{e}'_2$ for idempotents $\bar{e}'_1 \in \bar{S}_1$ and $\bar{e}'_2 \in \bar{S}_2$. Also, we have $\text{Ker}(\bar{h}_1) = (1 - \bar{e}_1)\bar{S}_1$ as $\bar{e}_1 \in \bar{S}_1$ is central and $(1 - e)h = 0$. This yields $\bar{e}_1 = \bar{e}'_1$.

Call $\bar{h}'_1 : \bar{S}_1 \rightarrow \bar{S}_1$ the homomorphism defined by $\bar{h}'_1|_{\bar{e}_1\bar{S}_1} = \bar{h}_1|_{\bar{e}_1\bar{S}_1}$ and $\bar{h}'_1|_{(1-\bar{e}_1)\bar{S}_1} = 1_{(1-\bar{e}_1)\bar{S}_1}$. By construction, \bar{h}'_1 is an automorphism in \bar{S}_1 . On the other hand, $\bar{h}_2 \in \bar{S}_2$, so we can write \bar{h}_2 as the sum of two automorphisms, say $\bar{h}_2 = \bar{h}_2' + \bar{h}_2''$. And again, \bar{h}_2'' can be written as the sum of two automorphisms in \bar{S}_2 , say $\bar{h}_2'' = \bar{t}_2 + \bar{t}_2'$.

Set $\bar{\gamma}_1 = \bar{h}'_1 \times \bar{h}_2'$, $\bar{\gamma}_2 = \bar{h}'_1 \times \bar{t}_2$, and $\bar{\gamma}_3 = (-\bar{h}'_1) \times \bar{t}_2'$. Consider then the homomorphism $\bar{\gamma} = \bar{\gamma}_1 + \bar{\gamma}_2 + \bar{\gamma}_3$. Then $\bar{\gamma}$ is the sum of three automorphisms $\bar{\gamma}_1$, $\bar{\gamma}_2$ and $\bar{\gamma}_3$ in \bar{S} . Note that for any $x_1 \times x_2 \in \bar{e}\bar{S} = \bar{e}_1\bar{S}_1 \times \bar{e}_2\bar{S}_2$, we have that $(\bar{h}'_1 \times \bar{h}_2' + \bar{h}'_1 \times \bar{t}_2 + (-\bar{h}'_1) \times \bar{t}_2')(x_1 \times x_2) = \bar{h}'_1(x_1) \times \bar{h}_2'(x_2) = \bar{h}_1(x_1) \times \bar{h}_2(x_2) = \bar{h}(x_1 \times x_2)$ since $\bar{h}'_1|_{\bar{e}_1\bar{S}_1} = \bar{h}_1|_{\bar{e}_1\bar{S}_1}$. This means that $\bar{\gamma}$ is the sum of three automorphisms in \bar{S} and $\bar{\gamma}|_{\bar{e}\bar{S}} = \bar{h}|_{\bar{e}\bar{S}}$. Let us lift the three automorphisms $\bar{\gamma}_i \in \bar{S}$ to automorphisms $\gamma_i \in S$. As M is automorphism-invariant, $\gamma_i(M) \subseteq M$ for $i = 1, 2, 3$. So if we call $\gamma = \gamma_1 + \gamma_2 + \gamma_3$, we get that $\gamma(M) \subseteq M$. Moreover, as $\bar{\gamma}|_{\bar{e}\bar{S}} = \bar{h}|_{\bar{e}\bar{S}}$, there exists a $j \in J(S)$ such that $\gamma|_{eS} = h|_{eS} + j|_{eS}$. Thus $h|_{eS} = (\gamma - j)|_{eS}$. As $j \in J(S)$, $1 - j$ is an automorphism and consequently, M is invariant under $1 - j$ and hence under j . We have already seen that M is invariant under γ . Thus it follows that M is invariant under $\gamma - j$. Call $\varphi = (\gamma - j)|_M$. Then φ is an endomorphism of S such that $\varphi|_{E(N)} = h|_{E(N)}$ and thus $\varphi|_N = f$. This shows that φ extends f and hence M is pseudo-injective.

The converse is straightforward (see [14]). \square

Now, we proceed to dualize this result for automorphism-coinvariant modules.

Recall that a module M is called a *pseudo-projective module* if for every submodule N of M , any epimorphism $\varphi : M \rightarrow M/N$ can be lifted to a homomorphism $\psi : M \rightarrow M$ (see [15]). These modules generalize the class of projective modules and quasi-projective modules (see [6, 13, 18]). It is known that any pseudo-projective module with a projective cover is dual automorphism-invariant (see [15]) and hence it is automorphism-coinvariant.

Theorem 2.3. *If R is a right perfect ring, then a right R -module M is automorphism-coinvariant if and only if it is pseudo-projective.*

Proof. Let M be an automorphism-coinvariant module over a right perfect ring R with a projective cover $p : P \rightarrow M$. Let N be a submodule of M and $f : M \rightarrow M/N$, an epimorphism. It is known that $\text{End}(P)/J(\text{End}(P))$ is von Neumann regular, right self-injective and idempotents lift modulo $J(\text{End}(P))$. As projective modules are clearly automorphism-coinvariant, by Theorem 2.1, we have that $P = P_1 \oplus P_2$, where $\text{Hom}(P_1, P_2), \text{Hom}(P_2, P_1) \subseteq J(\text{End}(P))$ and

$$\text{End}(P)/J(\text{End}(P)) = \text{End}(P_1)/J(\text{End}(P_1)) \times \text{End}(P_2)/J(\text{End}(P_2))$$

such that $\text{End}(P_1)/J(\text{End}(P_1))$ is a Boolean ring and each element in the ring $\text{End}(P_2)/J(\text{End}(P_2))$ is the sum of two units.

Let us denote by $\pi : M \rightarrow M/N$ the canonical projection and call $S = \text{End}(P)$, $S_1 = \text{End}(P_1)$ and $S_2 = \text{End}(P_2)$. As $p : P \rightarrow M$ is a projective cover, there exists a direct summand $P(M/N)$ of P such that, if we denote by $v : P(M/N) \rightarrow P$ and $q : P \rightarrow P(M/N)$ the structural injection and projection respectively, then $\pi \circ p \circ v : P(M/N) \rightarrow M/N$ is the projective cover of M/N , $e = v \circ q$ is an idempotent in S , and $P(M/N) = eP$.

By projectivity, f lifts to an epimorphism $g : P \rightarrow P(M/N)$ such that $(\pi \circ p \circ v) \circ g = f \circ p$. This epimorphism g splits as $P(M/N)$ is projective. Thus, there exists a monomorphism $\delta : P(M/N) \rightarrow P$ such that $g \circ \delta = 1_{P(M/N)}$. Call $h = v \circ g \in S$. Note that $e \circ h = h$ and so $hS \subseteq eS$. Moreover, $h \circ \delta \circ q = v \circ g \circ \delta \circ q = v \circ q = e$. So $eS \subseteq hS$ and consequently, $hS = eS$. This shows that $h : S \rightarrow eS$ is epic and consequently, $\bar{h} : \bar{S} \rightarrow \bar{e}\bar{S}$ is an epimorphism.

As $\bar{S} = \bar{S}_1 \times \bar{S}_2$, there exist idempotents $\bar{e}_1 \in \bar{S}_1$ and $\bar{e}_2 \in \bar{S}_2$ such that $\bar{e} = \bar{e}_1 \times \bar{e}_2 \in \bar{S}_1 \times \bar{S}_2$ and homomorphisms $\bar{h}_1 : \bar{S}_1 \rightarrow \bar{S}_1$ and $\bar{h}_2 : \bar{S}_2 \rightarrow \bar{S}_2$ such that $\bar{h} = \bar{h}_1 \times \bar{h}_2$.

Moreover, $\bar{e}_i \circ \bar{h}_i : \bar{S}_i \rightarrow \bar{e}_i \bar{S}_i$ is an epimorphism, and $(1 - \bar{e}_i) \circ \bar{h}_i = 0$ for $i = 1, 2$. As $\text{Im}(\bar{h}) \cong \bar{e}\bar{S}$, it is a direct summand of \bar{S} . So there exists an idempotent $\bar{e}' \in \bar{S}$ such that $\text{Im}(\bar{h}) = \bar{e}'\bar{S}$ and $\text{Ker}(\bar{h}) = (\bar{1} - \bar{e}')\bar{S}$. And again, $\bar{e}' = \bar{e}'_1 \times \bar{e}'_2$ for idempotents $\bar{e}'_1 \in \bar{S}_1$ and $\bar{e}'_2 \in \bar{S}_2$. Also, we have $\text{Ker}(\bar{h}_1) = (1 - \bar{e}_1)S$ as $\bar{e}_1 \in \bar{S}_1$ is central and $(1 - e)h = 0$. This yields $\bar{e}_1 = \bar{e}'_1$.

Call $\bar{h}'_1 : \bar{S}_1 \rightarrow \bar{S}_1$ the homomorphism defined by $\bar{h}'_1|_{\bar{e}_1 \bar{S}_1} = \bar{h}_1|_{\bar{e}_1 \bar{S}_1}$ and $\bar{h}'_1|_{(1 - \bar{e}_1) \bar{S}_1} = 1_{(1 - \bar{e}_1) \bar{S}_1}$. By construction, \bar{h}'_1 is an automorphism in \bar{S}_1 . On the other hand, $\bar{h}_2 \in \bar{S}_2$, so we can write \bar{h}_2 as the sum of two automorphisms, say $\bar{h}_2 = \bar{h}'_2 + \bar{h}''_2$. And again, \bar{h}''_2 can be written as the sum of two automorphisms in \bar{S}_2 , say $\bar{h}''_2 = \bar{t}_2 + \bar{t}'_2$.

Set $\bar{\gamma}_1 = \bar{h}'_1 \times \bar{h}'_2$, $\bar{\gamma}_2 = \bar{h}'_1 \times \bar{t}_2$, and $\bar{\gamma}_3 = (-\bar{h}'_1) \times \bar{t}'_2$. Consider then the homomorphism $\bar{\gamma} = \bar{\gamma}_1 + \bar{\gamma}_2 + \bar{\gamma}_3$. Then $\bar{\gamma}$ is the sum of three automorphisms $\bar{\gamma}_1$,

$\bar{\gamma}_2$ and $\bar{\gamma}_3$ in \bar{S} . Note that for any $x_1 \times x_2 \in \bar{e}\bar{S} = \bar{e}_1\bar{S} \times \bar{e}_2\bar{S}$, we have that $(h'_1 \times \bar{h}'_2 + \bar{h}'_1 \times \bar{t}_2 + (-\bar{h}'_1) \times \bar{t}'_2)(x_1 \times x_2) = \bar{h}'_1(x_1) \times \bar{h}_2(x_2) = \bar{h}_1(x_1) \times \bar{h}_2(x_2) = \bar{h}(x_1 \times x_2)$ since $\bar{h}'_1|_{\bar{e}_1\bar{S}} = \bar{h}_1|_{\bar{e}_1\bar{S}}$. This means that $\bar{\gamma}$ is the sum of three automorphisms in \bar{S} and $\bar{\gamma}|_{\bar{e}\bar{S}} = \bar{h}|_{\bar{e}\bar{S}}$. Let us lift the three automorphisms $\bar{\gamma}_i \in \bar{S}$ to automorphisms $\gamma_i \in S$. As M is automorphism-coinvariant, M is coinvariant under γ_i for $i = 1, 2, 3$. So if we call $\gamma = \gamma_1 + \gamma_2 + \gamma_3$, we get that M is coinvariant under γ . Moreover, as $\bar{e} \circ \bar{\gamma} = \bar{e} \circ \bar{h}$, there exists a $j \in J(S)$ such that $\pi \circ p \circ \gamma = \pi \circ p \circ h + \pi \circ p \circ j$. Thus $\pi \circ p \circ h = \pi \circ p \circ (\gamma - j)$. As $j \in J(S)$, $1 - j$ is an automorphism and consequently, M is coinvariant under $1 - j$ and hence under j . We have already seen that M is coinvariant under γ . Thus M is coinvariant under $\gamma - j$ and hence by definition, it follows that there exists an endomorphism $t : M \rightarrow M$ such that $p \circ (\gamma - j) = t \circ p$. This gives $\pi \circ p \circ h = \pi \circ t \circ p$. As $\pi \circ p \circ h = f \circ p$, we have $\pi \circ t \circ p = f \circ p$. As p is epic, we have $\pi \circ t = f$.

Thus t is an endomorphism of M such that $\pi \circ t = f$. This shows that M is pseudo-projective.

The converse is straightforward [15]. \square

Remark 2.4. *Note that the arguments given in the proof of Theorem 2.3 can be easily adapted to the situation in which the ring R is only assumed to be semiperfect instead of right perfect, and the module M is finitely generated and therefore, it has a projective cover.*

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REFERENCES

- [1] A. Alahmadi, A. Facchini, N. K. Tung, *Automorphism-invariant modules*, Rend. Sem. Mat. Univ. Padova, 133 (2015), 241-259.
- [2] F. W. Anderson, K. R. Fuller, *Rings and Categories of Modules*, Graduate Texts in Mathematics, 13, Springer-Verlag, New York, 1992.
- [3] J. Clark, C. Lomp, N. Vanaja, R. Wisbauer, *Lifting Modules: Supplements and projectivity in module theory*, Frontiers in Mathematics, Birkhauser Verlag, Basel, 2006.
- [4] S. E. Dickson, K. R. Fuller, *Algebras for which every indecomposable right module is invariant in its injective envelope*, Pacific J. Math. 31, 3 (1969), 655-658.
- [5] N. Er, S. Singh, A. K. Srivastava, *Rings and modules which are stable under automorphisms of their injective hulls*, J. Algebra 379 (2013), 223-229.
- [6] J. S. Golan, *Characterization of rings using quasi projective modules*, Israel J. Math. 8 (1970), 34-38.
- [7] P. A. Guil Asensio, D. Keskin Tütüncü and A. K. Srivastava, *Modules invariant under automorphisms of their covers and envelopes*, Israel J. Math. 206, 1 (2015), 457-482.
- [8] P. A. Guil Asensio, A. K. Srivastava, *Automorphism-invariant modules satisfy the exchange property*, J. Algebra 388 (2013), 101-106.
- [9] P. A. Guil Asensio, A. K. Srivastava, *Additive unit representations in endomorphism rings and an extension of a result of Dickson and Fuller*, Ring Theory and Its Applications, Contemporary Mathematics, Amer. Math. Soc. 609 (2014), 117-121.
- [10] P. A. Guil Asensio, A. K. Srivastava, *Automorphism-invariant modules*, Noncommutative rings and their applications, Contemporary Mathematics, Amer. Math. Soc., 634 (2015), 19-30.
- [11] P. A. Guil Asensio, T. C. Quynh, A. K. Srivastava, *Additive unit structure of endomorphism rings and invariance of modules under endomorphisms of their covers and envelopes*, preprint.

- [12] S. K. Jain, S. Singh, *On quasi-injective and pseudo-injective modules*, Canadian Math. Bull. 18 (1975), 359-366.
- [13] A. Koehler, *Quasi projective covers and direct sums*, Proc. Amer. Math. Soc. 24, 4 (1966), 655-658.
- [14] T. K. Lee, Y. Zhou, *Modules which are invariant under automorphisms of their injective hulls*, J. Algebra Appl. 12 (2), (2013).
- [15] S. Singh, A. K. Srivastava, *Dual automorphism-invariant modules*, J. Algebra 371 (2012), 262-275.
- [16] S. Singh, A. K. Srivastava, *Rings of invariant module type and automorphism-invariant modules*, Ring Theory and Its Applications, Contemporary Mathematics, Amer. Math. Soc. 609 (2014), 299-311.
- [17] M. Teply, *Pseudo-injective modules which are not quasi-injective*, Proc. Amer. Math. Soc. 49 (1975), 305-310.
- [18] L. E. T. Wu, J. P. Jans, *On quasi-projectives*, Illinois J. Math. 11 (1967), 439-448.

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